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Prepotential formulation of $SU(3)$ lattice gauge theory

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Abstract

The $SU(3)$ lattice gauge theory is reformulated in terms of $SU(3)$ prepotential harmonic oscillators. This reformulation has enlarged $SU(3) \otimes U(1) \otimes U(1)$ gauge invariance under which the prepotential operators transform like matter fields. The Hilbert space of $SU(3)$ lattice gauge theory is shown to be equivalent to the Hilbert space of the prepotential formulation satisfying certain color invariant $Sp(2, R)$ constraints. The $SU(3)$ irreducible prepotential operators which solve these $Sp(2, R)$ constraints are used to construct $SU(3)$ gauge invariant Hilbert spaces at every lattice site in terms of $SU(3)$ gauge invariant vertex operators. The electric fields and the link operators are reconstructed in terms of these $SU(3)$ irreducible prepotential operators. We show that all the $SU(3)$ Mandelstam constraints become local and take a very simple form within this approach. We also discuss the construction of all possible linearly independent $SU(3)$ loop states which solve the Mandelstam constraints. The techniques can be easily generalized to $SU(N)$.

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1. Introduction

The reformulation of gauge theories in terms of gauge invariant Wilson loops and strings carrying fluxes of the corresponding gauge group is an old problem in quantum field theory [1–6]. The motivation to go from colored gluons and quarks to colorless loops and string degrees of freedom comes from the expectation that the latter framework is better suited to analyze and understand long-distance non-perturbative issues like color confinement in QCD. In fact, the lattice formulation of gauge theories was a step in this direction where one directly works with link operators (instead of gauge connections) which create and destroy Abelian or non-Abelian loop fluxes on lattice links. However, the two major obstacles in this loop, string approach to QCD are the non-locality and proliferation of loops and string states [7]. The non-locality is obvious as the loops and strings can be of any shapes and sizes. The problem of proliferation exists because the set of all Wilson loop states forms a highly overcomplete basis. This is because not all loop states are mutually independent (see sections 3.3 and 4.6). Their relationships are expressed by the Mandelstam constraints. The Mandelstam constraints, in turn, are difficult to solve because of their non-locality (sections 3.3 and 4.6). Therefore, it is important to explore new descriptions of QCD where the loop, string states and their dynamics as well as the associated Mandelstam constraints can be analyzed locally. As shown in [8, 9], the prepotential approach to lattice gauge theories provides such a platform. More precisely, this approach allows us to analyze and solve the Mandelstam constraints locally at each lattice site without all the irrelevant non-local details associated with the loop states (sections 3.3 and 4.6). Toward this goal, a complete analysis was carried out for $SU(2)$ lattice gauge theory and all mutually independent loop states were constructed in terms of prepotential operators in [8, 9]. The purpose and motivation of this work is to analyze lattice QCD or $SU(3)$ lattice gauge theory within the prepotential framework. As we will see, there are many new issues which come up due to very different flux properties of $SU(3)$ and $SU(2)$ lattice gauge theories.

The prepotential operators are harmonic oscillators belonging to the fundamental representations of the gauge group. Further, unlike link operators which create and destroy fluxes on the links, the prepotential operators are associated with the sites and create or destroy smallest units of group fluxes at the corresponding lattice sites. In the case of $SU(2)$ lattice gauge theory [8], the prepotential approach enabled us to cast all the $SU(2)$ Mandelstam constraints in their local form. Further, all possible mutually orthonormal loop states were explicitly constructed in terms of the prepotential operators. The dynamics of these orthonormal $SU(2)$ loop states was shown to be governed by $3-nj$ Wigner coefficients. In fact, similar results have been obtained in the context of duality transformations in $SU(2)$ lattice gauge theories in [10–14]. More precisely, the $SU(2)$ gauge invariant basis labeled by (dual) angular momentum quantum numbers, describing two-dimensional triangulated surfaces, in [11] is exactly the same as the $SU(2)$ loop basis in [8] labeled by 'linking quantum numbers' which describe one-dimensional loops. In [6, 15–17] different computational schemes to identify independent $SU(2)$ loops were proposed. In [6, 15] loop Hamiltonians are computed in the above schemes retaining small loops carrying small fluxes[3]. In the context of loop quantum gravity, $SU(2)$ spin networks carrying $SU(2)$ fluxes which describe geometry of spacetime have been extensively studied [18, 19]. The $SU(2)$ Schwinger boson or equivalently prepotential techniques studied in [8, 9] can also be naturally applied to study the spin networks in loop quantum gravity as the fluxes in the spin networks are created by Schwinger bosons. This approach leads to many technical simplifications in the construction of spin networks and has been discussed extensively in [20]. On the other hand, in the context of QCD with the $SU(3)$ gauge group hardly any work has been done in these directions. In particular, it is important to construct and analyze all independent $SU(3)$ loop states ('$SU(3)$ spin networks) and study their dynamics. In the context of QCD, this analysis will be useful to analyze the spectrum of QCD Hamiltonian in terms of loops near the continuum limit where large loops carrying large fluxes are expected to dominate. The exact minimal loop basis containing arbitrarily large loops with all possible fluxes will allow us to analyze the spectrum without any spurious loop degrees of freedom.

In this work we show that the $SU(3)$ lattice gauge theory can also be completely described in terms of $SU(3)$ irreducible prepotentials with $SU(3)\otimes U(1)\otimes U(1)$ gauge invariance. Under $SU(3) \otimes U(1) \otimes U(1)$ gauge transformations the prepotentials transform like charged matter

[3] Note that the prepotential formulation resolves the issues of overcompleteness of $SU(2)$ loop states and their dynamics exactly without any assumptions.

fields. All the non-local $SU(3)$ Mandelstam constraints in terms of the link operators are cast into their local forms with the help of $SU(3)$ gauge invariant prepotential vertex operators which are defined at lattice sites (sections 3.3 and 4.6). We briefly discuss how to get all the solutions of $SU(3)$ Mandelstam constraints in the form of all possible independent $SU(3)$ loop states.

The paper is organized as follows. In section 2, we briefly discuss the Hamiltonian formulation of $SU(N)$ lattice gauge theory. This section sets up the notations and makes the paper self-contained. Section 3 starts with a brief summary of the $SU(2)$ prepotential approach to lattice gauge theory [8, 9]. This overview illustrates all the essential ideas involved in simplifying the Mandelstam constraints and getting all their solutions in the simpler $SU(2)$ case before dealing with their more involved $SU(3)$ analogs. In addition, this section also helps us to highlight some completely new issues and difficulties one confronts on going from $SU(2)$ to $SU(3)$ lattice gauge theory. Section 4 discusses $SU(3)$ lattice gauge theory in terms of prepotential operators. In sections 4.1 and 4.2 we study and classify the $SU(3)$ prepotential Hilbert space $\mathcal{H}_p$ according to $SU(3)$ invariant $Sp(2, R)$ quantum numbers [21]. In section 4.3, we show that the Hilbert space of $SU(3)$ gauge theory $\mathcal{H}_g$ is a tiny subspace of $\mathcal{H}_p$ which satisfies certain $Sp(2, R)$ constraints. Section 4.4 deals with $SU(3)$ irreducible prepotential operators [22] which are solutions of the above $Sp(2, R)$ constraints and therefore directly create the gauge theory Hilbert space $\mathcal{H}_g$. The explicit construction of $SU(3)$ link operators and electric fields in terms of the $SU(3)$ irreducible prepotentials is given in section 4.5. In section 4.6, with the help of $SU(3)$ irreducible prepotential operators, we construct all possible $SU(3)$ gauge invariant vertices at a given lattice site which in turn cast all $SU(3)$ Mandelstam constraints in their local forms. Having made them local, section 4.6.1 discusses how to solve these infinite sets of constraints at every lattice site exactly. We then briefly discuss the prepotential formulation of $SU(N)$ lattice gauge theory. We end the paper with a brief summary and discussion on related issues.

2. *SU*(*N*) Hamiltonian formulation

The Hamiltonian of $SU(N)$ lattice gauge theory is

$$H = \sum_{n,i} \sum_{\mathrm{a}=1}^{N^2-1} E^{\mathrm{a}}(n,i)E^{\mathrm{a}}(n,i) + K \sum_{\text{plaquette}} \mathrm{Tr}\big(U_{\text{plaquette}} + U^{\dagger}_{\text{plaquette}}\big) \tag{1}$$

with

$$U_{\text{plaquette}} = U(n,i)U(n+i,j)U^{\dagger}(n+j,i)U^{\dagger}(n,j),$$

where K is the coupling constant, a$(=1, 2, \ldots, (N^2-1))$ is the color index. In (1) the kinematical operators E and U can be understood as follows. Each link (n, i) is associated with an $SU(N)$ symmetric top whose configuration (i.e. the rotation matrix from space fixed to body fixed frame) is given by the operator-valued $(N \times N)$ $SU(N)$ matrix $U(n, i)$. Let $E^{\mathrm{a}}_L(n,i)$, $E^{\mathrm{a}}_R(n+i,i)$ denote the conjugate left and right electric fields with the quantization rules [5]:

$$\big[E^{\mathrm{a}}_L(n,i), U^{\alpha}{}_{\beta}(n,i)\big] = -(T^{\mathrm{a}}U(n,i))^{\alpha}{}_{\beta}, \qquad \big[E^{\mathrm{a}}_R(n+i,i), U^{\alpha}_{\beta}(n,i)\big] = (U(n,i)T^{\mathrm{a}})^{\alpha}{}_{\beta}. \tag{2}$$

In (2), T^{a} are the generators in the fundamental representation of $SU(N)$ and satisfy $[T^{\mathrm{a}}, T^{\mathrm{b}}] = if^{\mathrm{abc}}T_{\mathrm{c}}$ where f^{abc} are the $SU(N)$ structure constants. The quantization rules (2) clearly show that $E_L(n,i)$ and $E_R(n+i,i)$ are the generators of left and the right gauge transformations in (7). In fact, the right generators $E^{\mathrm{a}}_R(n+i,i)$ are the parallel transport of the left generator $E^{\mathrm{a}}_L(n,i)$ on the link (n,i):

$$E_R(n+i,i) = -U^{\dagger}(n,i)E_L(n,i)U(n,i). \tag{3}$$

In (3), $E_R(n+i,i) \equiv \sum_{\mathrm{a}} E^{\mathrm{a}}_R(n+i,i)T^{\mathrm{a}}$ and $E_L(n,i) \equiv \sum_{\mathrm{a}} E^{\mathrm{a}}_L(n,i)T^{\mathrm{a}}$. The left and the right electric fields on every link, being the *SU*(*N*) rotation generators, satisfy

$$\left[E^{\mathrm{a}}_L(n,i), E^{\mathrm{b}}_L(n,i)\right] = i f_{\mathrm{abc}} E^{\mathrm{c}}_L(n,i), \qquad \left[E^{\mathrm{a}}_R(n,i), E^{\mathrm{b}}_R(n,i)\right] = i f_{\mathrm{abc}} E^{\mathrm{c}}_R(n,i). \tag{4}$$

Further, using (3), it is easy to show that E^{a}_L and E^{a}_R commute among themselves:

$$\left[E^{\mathrm{a}}_L(n,i), E^{\mathrm{b}}_R(m,j)\right] = 0 \tag{5}$$

and therefore mutually independent. By construction on each link they always satisfy the constraints

$$\sum_{\mathrm{a}=1}^{N^2-1} E^{\mathrm{a}}(n,i)E^{\mathrm{a}}(n,i) \equiv \sum_{\mathrm{a}=1}^{N^2-1} E^{\mathrm{a}}_L(n,i)E^{\mathrm{a}}_L(n,i) = \sum_{\mathrm{a}=1}^{N^2-1} E^{\mathrm{a}}_R(n+i,i)E^{\mathrm{a}}_R(n+i,i). \tag{6}$$

The Hamiltonian in (1) involves the squares of either left or the right electric fields. Under gauge transformation the left electric field and the link operator transform as

$$\begin{aligned} &U(n,i) \to \Lambda(n)U(n,i)\Lambda^{\dagger}(n+i), \\ &E_L(n,i) \to \Lambda(n)E_L(n,i)\Lambda^{\dagger}(n), \qquad E_R(n+i,i) \to \Lambda(n+i)E_R(n+i,i)\Lambda^{\dagger}(n+i). \end{aligned} \tag{7}$$

The Hamiltonian (1) and the basic commutation relations (2) are invariant under the *SU*(*N*) gauge transformations (7). From (7), the $SU(N)$ Gauss law constraint at every lattice site *n* is

$$G(n) = \sum_{i=1}^{d} \left(E^{\mathrm{a}}_L(n,i) + E^{\mathrm{a}}_R(n,i)\right) = 0, \qquad \forall\, n. \tag{8}$$

It is convenient to define the left and right strong coupling vacuum states $|0\rangle_L$ and $|0\rangle_R$ on every link which are annihilated by their corresponding electric fields:

$$E^{\mathrm{a}}_L(n,i)|0,(n,i)\rangle_L = 0, \qquad E^{\mathrm{a}}_R(n+i,i)|0,(n+i,i)\rangle_R = 0, \forall \text{ links } (n,i). \tag{9}$$

We will denote the vacuum state on a link by $|0\rangle \equiv |0,(n,i)\rangle_L \otimes |0,(n,i)\rangle_R$, suppressing all the link as well as L, R indices. The quantization rules (2) show that the link operators $U^{\alpha}{}_{\beta}(n,i)$ acting on the strong coupling vacuum (9) create *SU*(*N*) fluxes on the links. As an example, using (2):

$$E^2_L(n,i)\left(U^{\alpha}{}_{\beta}|0\rangle\right) = E^2_R(n+i,i)\left(U^{\alpha}{}_{\beta}|0\rangle\right) = \frac{1}{2N}(N^2-1)\left(U^{\alpha}{}_{\beta}|0\rangle\right). \tag{10}$$

The higher *SU*(3) irreducible flux eigenstates of E^2_L and E^2_R on a link can be obtained by considering the states $U^{\alpha_1}{}_{\beta_1}U^{\alpha_2}{}_{\beta_2}\cdots U^{\alpha_1}{}_{\beta_1}|0\rangle$ and symmetrizing α and therefore also β indices according to certain *SU*(*N*) Young tableau. We will discuss this issue again in section 3 and section 4 in the specific context of *SU*(2) and *SU*(3) groups.

3. Prepotentials in *SU*(2) lattice gauge theory

In this section we define *SU*(2) prepotential operators. Using the Schwinger bosons construction of the angular momentum algebra (4), the left and the right electric fields on a link (n,i) can be written as

$$\begin{aligned} &\text{left electric fields:} \qquad E^{\mathrm{a}}_L(n,i) \equiv a^{\dagger}(n,i;L)\frac{\sigma^{\mathrm{a}}}{2}a(n,i;L), \\ &\text{right electric fields:} \quad E^{\mathrm{a}}_R(n+i,i) \equiv a^{\dagger}(n+i,i;R)\frac{\sigma^{\mathrm{a}}}{2}a(n+i,i;R). \end{aligned} \tag{11}$$

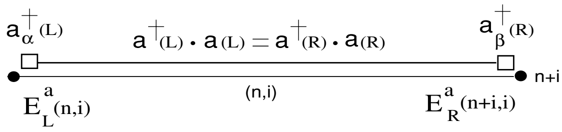

Figure 1. The left and right electric fields and the corresponding prepotentials in $SU(2)$ lattice gauge theory. We have denoted $a^\dagger(n,i,L)$ and $a^\dagger(n+i,i,R)$ by $a^\dagger(L)$ and $a^\dagger(R)$ respectively. The unoriented Abelian flux line connecting them represents the $U(1)$ Gauss law (18) constraint.

In (11), $a_\alpha(n,i;l)$ and $a^\dagger_\alpha(n,i;l)$ are the doublets of harmonic oscillator creation and annihilation operators with $l = L, R, \alpha = 1, 2$. We have used Schwinger boson construction [23] of angular momentum algebra in (11). Like $E^{\mathrm{a}}_L(n,i)$ and $E^{\mathrm{a}}_R(n+i,i)$, the locations of $a(n,i,L), a^\dagger(n,i,L)$ and $a(n+i,i,R), a^\dagger(n+i,i,R)$ are on the left and the right of the link (n,i). For notational convenience we suppress the link indices and denote $a^\dagger(n,i,L)$ and $a^\dagger(n+i,i,R)$ by $a^\dagger(L)$ and $a^\dagger(R)$ respectively. This is clearly illustrated in figure 1. The link indices will be explicitly shown whenever we work with more than one link. Note that relations (11) imply that the strong coupling vacuum (9) is the harmonic oscillator vacuum. Substituting the electric fields (11) in terms of Schwinger bosons in the electric field constraints (6), we get $a^\dagger(n,i;L)\cdot a(n,i;L) = a^\dagger(n+i,i;R)\cdot a(n+i,i;R)$. We will come back to this issue again in section 3.1.

Under $SU(2)$ gauge transformation with the generator $G(n)$ in (8), the prepotential harmonic oscillator transforms as $SU(2)$ doublets[4]:

$$\begin{aligned} a^\dagger_\alpha(L) &\to a^\dagger_\beta(L)\big(\Lambda^\dagger_L\big)^\beta{}_\alpha, & a^\dagger_\alpha(R) &\to a^\dagger_\beta(R)\big(\Lambda^\dagger_R\big)^\beta{}_\alpha \\ a^\alpha(L) &\to (\Lambda_L)^\alpha{}_\beta a^\beta(L), & a^\alpha(R) &\to (\Lambda_R)^\alpha{}_\beta a^\beta(R). \end{aligned} \tag{12}$$

One can also define $\tilde{a}^{\dagger\alpha} = \epsilon^{\alpha\beta}a^\dagger_\beta$ and $\tilde{a}_\alpha = \epsilon_{\alpha\beta}a^\beta$ which under $SU(2)$ transformation transform as a^α and $a^\dagger_\alpha$ respectively. In terms of link operators the basic $SU(2)$ flux states on links can be constructed using the link operators:

$$|j(n,i), m_L(n,i), m_R(n,i)\rangle = \big(U^{\alpha_1}{}_{\beta_1}U^{\alpha_2}{}_{\beta_2}\cdots U^{\alpha_{2j}}{}_{\beta_{2j}} + \cdots (2j)!\ \text{permutations}\big)|0\rangle. \tag{13}$$

In (13), $j_L(n,i) = j_R(n+i,i) \equiv j(n,i)$ because of (6), $m_L = \sum_{i=1}^{2j}\alpha_i$ and $m_R = \sum_{i=1}^{2j}\beta_i$ with $\alpha_i, \beta_i = \pm\frac{1}{2}$. The $(2j)!$ terms in (13) are required to implement the symmetries of $SU(2)$ Young tableau in the left $(\alpha_1\alpha_2\cdots\alpha_{2j})$ as well as the right $(\beta_1\beta_2\cdots\beta_{2j})$ indices. The gauge theory Hilbert space $\mathcal{H}_g$ is spanned by direct product of states of type (13) on all the lattice links. Note that as the flux value $j\to\infty$ on various links[5], the construction of the gauge theory Hilbert space $\mathcal{H}_g$ through (13) becomes more and more tedious. The basic link states in (13) can now be disentangled into its left and right parts as

$$|j(n,i), m_L(n,i), m_R(n,i)\rangle = |j(n,i), m_L(n,i)\rangle_L \otimes |j(n,i), m_R(n,i)\rangle_R, \tag{14}$$

where

$$\begin{aligned} |j(n,i), m_L(n,i)\rangle_L &= a^\dagger_{\alpha_1}(L)a^\dagger_{\alpha_2}(L)\cdots a^\dagger_{\alpha_n}(L)|0\rangle_L \equiv \hat{\mathcal{L}}_{\alpha_1\alpha_2\cdots\alpha_n}|0\rangle_L, \\ |j(n,i), m_R(n,i)\rangle_R &= a^\dagger_{\beta_1}(R)a^\dagger_{\beta_2}(R)\cdots a^\dagger_{\beta_n}(R)|0\rangle_R \equiv \hat{\mathcal{R}}_{\beta_1\beta_2\cdots\beta_n}|0\rangle_R. \end{aligned} \tag{15}$$

In (15), $n = 2j, m_L = \sum_{i=1}^{2j}\alpha_i$ and $m_R = \sum_{i=1}^{2j}\beta_i$ with $\alpha_i, \beta_i = \pm\frac{1}{2}$. The operators $\hat{\mathcal{L}}$ and $\hat{\mathcal{R}}$ are the $SU(2)\otimes U(1)$ flux creation operators at the left and right ends of every link. Note that

[4] Here we specify the notations in (12): $a^\dagger_\alpha(L) \equiv a^\dagger_\alpha(n,i;L), a^\dagger_\alpha(R) \equiv a^\dagger_\alpha(n+i,i;R)$ are located on the left and right sides of the link (n,i) and $\Lambda_L \equiv \Lambda(n), \Lambda_R \equiv \Lambda(n+i)$ as shown in figure 1 explicitly.

[5] These large j configurations are expected to dominate in the continuum $(g\to 0)$ limit.

these operators are $SU(2)$ irreducible as they are symmetric in all the $SU(2)$ spin half indices and are defined for later convenience (see section 4.2). From (13) and (15) we conclude that the Hilbert space $\mathcal{H}_p$ created using the prepotential operators on all lattice links is also the $SU(2)$ gauge theory Hilbert space:

$$\mathcal{H}_g \equiv \mathcal{H}_p. \tag{16}$$

However, the construction of $\mathcal{H}_g$ using the prepotentials (15) is much simpler than the equivalent equivalent construction (13) using the link operators. This simplicity occurs because unlike the link operators $U_{\alpha\beta}(n,i)$ which are associated with links, the prepotential operators are attached to the sites (i.e. left or right ends of every link). Further, all the $SU(2)$ prepotential creation operators commute among themselves and we do not need $(2j)!$ terms (as in (13)) to get the symmetries of $SU(2)$ Young tableau. In other words, the symmetries of $SU(2)$ Young tableau are inbuilt in $SU(2)$ prepotential operators. We will come back to this symmetry issue (end of section 4.3) and the identification of $\mathcal{H}_g$ with $\mathcal{H}_p$ (16) (see equations (35) and (46)) again when we discuss $SU(3)$ lattice gauge theory in terms of prepotential operators.

3.1. $U(1)$ gauge invariance

The defining equations for the prepotential operators are invariant under $U(1) \otimes U(1)$ gauge transformations on every link:

$$a^\dagger_\alpha(L) \rightarrow \mathrm{e}^{\mathrm{i}\theta(L)} a^\dagger_\alpha(L), \qquad a^\dagger_\alpha(R) \rightarrow \mathrm{e}^{-\mathrm{i}\theta(R)} a^\dagger_\alpha(R). \tag{17}$$

Note that the above Abelian gauge transformations are defined on the two sides of every link and are independent of the $SU(2)$ gauge transformations (12) which are defined at every lattice site. Using (11), the electric field constraints (6) on the links become the number operator constraints in terms of the prepotential operators:

$$\hat{N}(L) \equiv a^\dagger(L)\cdot a(L) = \hat{N}(R) \equiv a^\dagger(R)\cdot a(R) \equiv \hat{N}. \tag{18}$$

In (18), $\hat{N} \equiv \hat{N}(n,i)$ and imply $\theta(L) = \theta(R)$ on every link and reduces the extra $U(1)\otimes U(1)$ gauge invariance to $U(1)$. Thus in the prepotential formulation non-Abelian fluxes can be absorbed locally at a site and the Abelian fluxes spread along the links. Both the gauge symmetries together lead to non-local (involving at least a plaquette) Wilson loop states (see section 3.3).

3.2. SU(2) link operators

Equations (11) already define the left and right electric fields in terms of the prepotentials. To establish complete equivalence, we now write down the link operators explicitly in terms of the prepotentials. From $SU(2)$ gauge transformations of the link operator in (7) and $SU(2)\otimes U(1)$ gauge transformations (12), (17) of the prepotentials,

$$U^\alpha{}_\beta = \tilde{a}^{\dagger\alpha}(L)\eta a^\dagger_\beta(R) + a^\alpha(L)\theta\tilde{a}_\beta(R), \tag{19}$$

where η and θ are functions of $SU(2)$ invariant number operator. The operators $\tilde{a}^{\dagger\alpha}$ and $\tilde{a}_\beta$ are defined after equation (12). Equation (19) is graphically illustrated in terms of $SU(2)$ Young tableaus in figure 2.

In the explicit matrix form the link operator can be written as the product of the left part U_L and the right part U_R as

$$U = \underbrace{\begin{pmatrix} a^\dagger_2(L)\eta_L & a_1(L)\theta_L \\ -a^\dagger_1(L)\eta_L & a_2(L)\theta_L \end{pmatrix}}_{U_L} \underbrace{\begin{pmatrix} \eta_R a^\dagger_1(R) & \eta_R a^\dagger_2(R) \\ \theta_R a_2(R) & \theta_R(-a_1(R)) \end{pmatrix}}_{U_R} \tag{20}$$

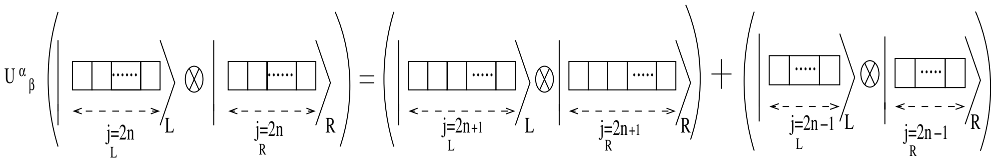

Figure 2. The Young tableau interpretation of the $SU(2)$ link operator U in terms of the prepotential operators (19) acting on a state with $n_L = n_R = 2j$. The two terms in (19) correspond to the two sets of Young tableaus on the right-hand side of this figure respectively.

where, $\eta_L, \eta_R, \theta_L, \theta_R$ are the left and right invariants constructed out of number operators. From (19) it follows that $\eta = \eta_L\eta_R, \theta = \theta_L\theta_R$. From (20),

$$U_L^\dagger U_L = \begin{pmatrix} \bar{\eta}_L[a^\dagger(L)\cdot a(L)+2]\eta_L & 0 \\ 0 & \bar{\theta}_L[a^\dagger(L)\cdot a(L)]\theta_L \end{pmatrix},$$
$$U_R U_R^\dagger = \begin{pmatrix} \eta_R[a^\dagger(R)\cdot a(R)]\bar{\eta}_R & 0 \\ 0 & \theta_R[a^\dagger(R)\cdot a(R)+2]\bar{\theta}_R \end{pmatrix}. \tag{21}$$

Therefore, for $U_{\alpha\beta}$ to be unitary we get

$$\eta_L = \frac{1}{\sqrt{a^\dagger(L)\cdot a(L)+2}}, \qquad \theta_L = \frac{1}{\sqrt{a^\dagger(L)\cdot a(L)}},$$
$$\eta_R = \frac{1}{\sqrt{a^\dagger(R)\cdot a(R)}}, \qquad \theta_R = \frac{1}{\sqrt{a^\dagger(R)\cdot a(R)+2}}. \tag{22}$$

Note that the operator η_R above is always well defined as it always appears with $a^\dagger_\alpha(R)$ on its right in (20). The operator θ_L is well defined in (20) as the link operator $U \equiv U_L U_R$ acts on the Hilbert space satisfying the constraints (18). Finally, using $a^\dagger(L)\cdot a(L) = a^\dagger(R)\cdot a(R) \equiv \hat{N}$, the link operator can be disentangled into its left and right parts as

$$U = \underbrace{\frac{1}{\sqrt{\hat{N}+1}}\begin{pmatrix} a_2^\dagger(L) & a_1(L) \\ -a_1^\dagger(L) & a_2(L) \end{pmatrix}}_{U_L} \underbrace{\begin{pmatrix} a_1^\dagger(R) & a_2^\dagger(R) \\ a_2(R) & -a_1(R) \end{pmatrix}\frac{1}{\sqrt{\hat{N}+1}}}_{U_R} \equiv U_L U_R \tag{23}$$

and satisfies $U^\dagger U = UU^\dagger = 1$.

3.3. SU(2) gauge invariant states and Mandelstam constraints

The prepotential operators being associated with sites enable us to construct $SU(2)$ gauge invariant Hilbert spaces at every lattice site. These $SU(2)$ gauge invariant Hilbert spaces at different lattice sites are mutually orthogonal. Therefore, the Mandelstam constraints which relate the various gauge invariant states can be analyzed and solved locally at each lattice site. For a d-dimensional lattice we have $2d$ number of prepotential creation operators present at each site all transforming in the same way under the $SU(2)$ group present at the site (see figure 3). Hence, all possible $SU(2)$ invariant creation operators at site n are constructed by anti-symmetrizing any two different prepotential doublets:

$$L_{ij}(n) = \epsilon^{\alpha\beta}a^\dagger_\alpha(n,i)a^\dagger_\beta(n,j) = a^\dagger(n,i)\cdot\tilde{a}^\dagger(n,j), \qquad i,j = 1,2,\ldots,2d. \tag{24}$$

In (24), $a^\dagger_\alpha(n,i)$ with $i = 1,2,\ldots 2d$ denote the $2d$ prepotentials around the lattice site n (see figure 3 for $d = 2$). Hence, the most general gauge invariant states at a lattice site n are given

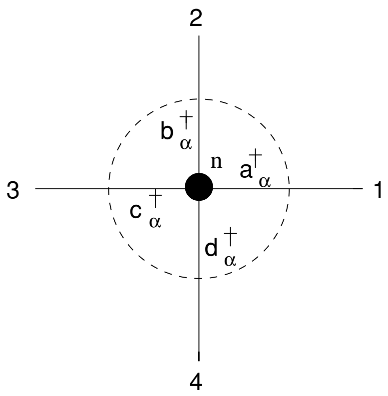

Figure 3. $SU(2)$ prepotentials associated with a lattice site n in a $d = 2$ lattice. A $SU(2)$ gauge transformation at site n affects only these prepotentials enabling us to construct $SU(2)$ gauge invariant Hilbert spaces locally at each lattice site.

by

$$|\vec{l}(n)\rangle = \prod_{i,j=1}^{2d} (L_{ij}(n))^{l_{ij}(n)}|0\rangle. \tag{25}$$

But these $|\vec{l}(n)\rangle$ states form an overcomplete basis because of the Mandelstam constraints[6] [8]:

$$(a^{\dagger} \cdot \tilde{b}^{\dagger})(c^{\dagger} \cdot \tilde{d}^{\dagger}) \equiv (a^{\dagger} \cdot \tilde{c}^{\dagger})(b^{\dagger} \cdot \tilde{d}^{\dagger}) - (a^{\dagger} \cdot \tilde{d}^{\dagger})(b^{\dagger} \cdot \tilde{c}^{\dagger}). \tag{26}$$

A complete orthonormal gauge invariant basis at site n in terms of $SU(2)$ prepotentials is given in terms of $SU(2)$ angular momentum quantum numbers [8]:

$$|LS\rangle_n \equiv |j_1, j_2, \ldots j_{2d}; j_{12}, j_{123}, \ldots j_{12\ldots(2d-1)} = j_{2d}\rangle = N(j){\sum_{\{l\}}}' \prod_{\substack{i,j \\ i<j}} \frac{1}{l_{ij}!}\big(L_{ij}(n)\big)^{l_{ij}(n)}|0\rangle. \tag{27}$$

The prime over the summation means that the linking numbers l_{ij} are summed over all possible values which are consistent with certain geometrical constraints [8]. The states (27) at different lattice sites along with $U(1)$ constraints (18) describe all possible orthonormal (linearly independent) loop states. It is also shown [8] that the loop dynamics for pure $SU(2)$ lattice gauge theory in d dimension is given by real and symmetric $3nj$ Wigner coefficients of the second kind (e.g. $n = 6, 10$ for $d = 2, 3$ respectively).

4. Prepotentials in $SU(3)$ lattice gauge theory

We will now generalize the above $SU(2)$ prepotential formulation to $SU(3)$ lattice gauge theory. Like in $SU(2)$, the $SU(3)$ prepotentials are defined through the left and right electric fields in $SU(3)$ lattice gauge theory. However, now the two fundamental representations 3 (quark) and 3^* (anti-quark) of $SU(3)$ are independent. Hence, we associate two independent harmonic oscillator prepotential triplets:

$$a^{\dagger}_{\alpha}(n, i; L) \equiv a^{\dagger}_{\alpha}(L), \qquad b^{\dagger\alpha}(n, i; L) \equiv b^{\dagger\alpha}(L), \qquad \alpha = 1, 2, 3,$$

[6] We will discuss the Mandelstam constraints in detail in section 4.4.

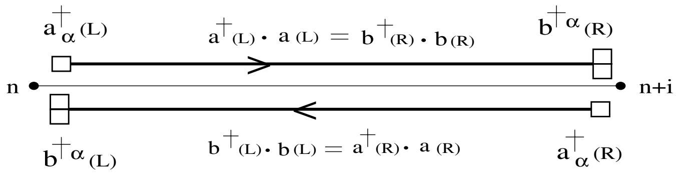

Figure 4. The $SU(3)$ prepotentials and the two $U(1)\otimes U(1)$ oriented Abelian flux lines along a link in $SU(3)$ lattice gauge theory. The directions of Abelian flux lines are chosen from quark ($a^\dagger$) prepotentials to anti-quark ($b^\dagger$) prepotentials.

to the left end and

$$a^\dagger_\alpha(n+i,i;R) \equiv a^\dagger_\alpha(R), \qquad b^{\dagger\alpha}(n+i,i;R) \equiv b^{\dagger\alpha}(R), \qquad \alpha = 1,2,3,$$

to the right end of the link (n,i). Now there are 12 prepotential operators associated with every link. These assignments are shown in figure 4. Under $SU(3)$ gauge transformation in a d-dimensional spatial lattice, the $2d$ $a^\dagger s$ and $2d$ $b^\dagger s$ on the $2d$ links emanating from the lattice site n transform as quarks (3) and anti-quarks (3^*) respectively. The $SU(3)$ electric fields are

$$\begin{aligned}
&\text{left electric fields:} \quad E^{\mathrm{a}}_L = \left(a^\dagger(L)\frac{\lambda^{\mathrm{a}}}{2}a(L) - b(L)\frac{\lambda^{\mathrm{a}}}{2}b^\dagger(L)\right)\\
&\text{right electric fields:} \quad E^{\mathrm{a}}_R = \left(a^\dagger(R)\frac{\lambda^{\mathrm{a}}}{2}a(R) - b(R)\frac{\lambda^{\mathrm{a}}}{2}b^\dagger(R)\right).
\end{aligned} \tag{28}$$

In (28), we have used Schwinger boson construction of $SU(3)$ Lie algebra [24, 25]. The electric field generators in (28) generate $SU_L(3)\otimes SU_R(3)$ gauge transformations on every link. The prepotential triplets satisfy the standard harmonic oscillator commutation relations:

$$\begin{aligned}
&\left[a^\alpha(l), a^\dagger_\beta(l')\right] = \delta^\alpha_\beta\delta_{l,l'}, \qquad [b_\alpha(l), b^{\dagger\beta}(l')] = \delta^\beta_\alpha\delta_{l,l'},\\
&[a^\alpha(l), a^\beta(l')] = 0, \qquad [b_\alpha(l), b_\beta(l')] = 0, \qquad l,l' = L,R.
\end{aligned} \tag{29}$$

As all the electric fields in (28) involve both creation and annihilation operators, the number operators in (30) commute with all the electric fields in (28). Therefore, the two $SU(3)$ Casimirs on each side of the link (n, i) are

$$\begin{aligned}
&\hat{N}(L) = a^\dagger(L)\cdot a(L), \qquad \hat{N}(R) = a^\dagger(R)\cdot a(R),\\
&\hat{M}(L) = b^\dagger(L)\cdot b(L), \qquad \hat{M}(R) = b^\dagger(R)\cdot b(R).
\end{aligned} \tag{30}$$

The eigenvalues of $\hat{N}(L), \hat{M}(L)$ and $\hat{N}(R), \hat{M}(R)$ will be denoted by n_L, m_L and n_R, m_R respectively. We can characterize all the $SU(3)$ irreducible representations on a link by $(n_L, m_L)\otimes(n_R, m_R)$. Using the Gauss law generators (8) and the defining equations (4), the $SU(3)$ gauge transformations of the prepotentials on the left and right sides of a link (n, i) are

$$\begin{aligned}
&a^\dagger_\alpha(L) \to a^\dagger_\beta(L)\left(\Lambda^\dagger_L\right)^\beta{}_\alpha, \qquad a^\dagger_\alpha(R) \to a^\dagger_\beta(R)\left(\Lambda^\dagger_R\right)^\beta{}_\alpha,\\
&b^{\dagger\alpha}(L) \to (\Lambda_L)^\alpha{}_\beta b^{\dagger\beta}(L), \qquad b^{\dagger\alpha}(R) \to (\Lambda_R)^\alpha{}_\beta b^{\dagger\beta}(R).
\end{aligned} \tag{31}$$

The above transformations imply that under $SU(3)$ gauge transformations $a^\dagger_\alpha(L), a^\dagger_\alpha(R)$ transform like quarks and $b^{\dagger\alpha}(L), b^{\dagger\alpha}(R)$ transform like anti-quarks at the left and the right end of the link (n, i) respectively. Therefore, we call $a, a^\dagger$ and $b, b^\dagger$ on various links as quark and anti-quark prepotentials respectively.

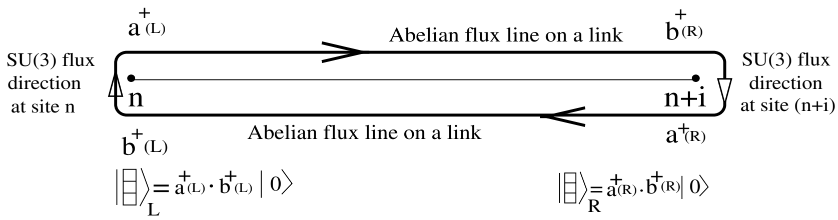

Figure 5. The graphical interpretation of the $SU(3) \otimes U(1) \otimes U(1)$ gauge invariant loop state (36) over a link (n, i) with $n_L = n_R = n = 1$. The 'magnetic' $Sp(2, R)$ quantum number ρ of this state is non-zero ($\rho = 1$) and therefore such states cannot be created by the link operators $U(n, i)$. Two types of arrows are used to differentiate Abelian and non-Abelian fluxes.

4.1. The $U(1) \otimes U(1)$ gauge invariance

Like in the $SU(2)$ case (see (17)), the defining equations of $SU(3)$ prepotentials (28) are invariant under the following $U(1) \otimes U(1) \otimes U(1) \otimes U(1)$ Abelian gauge transformations:

$$\begin{aligned} a^\dagger_\alpha(L) &\to \mathrm{e}^{\mathrm{i}\theta(L)} a^\dagger_\alpha(L), & a^\dagger_\alpha(R) &\to \mathrm{e}^{-\mathrm{i}\phi(R)} a^\dagger_\alpha(R), \\ b^{\dagger\alpha}(L) &\to \mathrm{e}^{\mathrm{i}\phi(L)} b^{\dagger\alpha}(L), & b^{\dagger\alpha}(R) &\to \mathrm{e}^{-\mathrm{i}\theta(R)} b^{\dagger\alpha}(R). \end{aligned} \tag{32}$$

In (32), the Abelian gauge angles $\theta(l)$ and $\phi(l)$ with $l = L, R$ are defined on the left and right sides of every link. Again like in the $SU(2)$ case, the Hilbert space of lattice gauge theory is built by applying the link operators on the vacuum state:

$$U^{\alpha_1}{}_{\beta_1} U^{\alpha_2}{}_{\beta_2} \cdots U^{\alpha_n}{}_{\beta_n} |0\rangle$$

and then symmetrizing/anti-symmetrizing αs according to a certain Young tableau. However, this symmetrizing/anti-symmetrizing the left $\alpha \in 3$ indices automatically induces the same symmetries/anti-symmetries on the right $\beta \in 3^*$ indices. This implies that the left and right representations are always conjugate to each other[7], i.e.

$$\hat{N}(L) = \hat{M}(R), \qquad \hat{M}(L) = \hat{N}(R). \tag{33}$$

This implies $\theta(L) = \theta(R)$ and $\phi(L) = \phi(R)$ on every link. Therefore, besides $SU(3)$ gauge invariance (31) at different lattice sites, the prepotential formulation has additional Abelian $U(1) \otimes U(1)$ gauge invariance (32) on every link. The Gauss law constraints (33) imply that Abelian fluxes are oriented. We choose the directions of the Abelian fluxes on links to be from quark to anti-quark prepotentials. To maintain continuity of direction in a loop state the non-Abelian fluxes are chosen in the opposite direction (i.e. from anti-quark prepotentials to quark prepotentials). These conventions are clearly illustrated on a link in figures 4 and 5.

4.2. The SU(3) prepotential Hilbert space $\mathcal{H}_p$

Like in the $SU(2)$ case (15), the Hilbert space of $SU(3)$ prepotential operators $\mathcal{H}_p$ can be completely characterized by the following basis on every lattice link:

$$\left|{}^{\beta_1\beta_2\cdots\beta_q}_{\alpha_1\alpha_2\cdots\alpha_p}\right\rangle_L \otimes \left|{}^{\delta_1\delta_2\cdots\delta_p}_{\gamma_1\gamma_2\cdots\gamma_q}\right\rangle_R \equiv \hat{L}^{\beta_1\beta_2\cdots\beta_q}_{\alpha_1\alpha_2\cdots\alpha_p}|0\rangle_L \otimes \hat{R}^{\delta_1\delta_2\cdots\delta_p}_{\gamma_1\gamma_2\cdots\gamma_q}|0\rangle_R. \tag{34}$$

In (34),

$$\hat{L}^{\beta_1\beta_2\cdots\beta_q}_{\alpha_1\alpha_2\cdots\alpha_p}|0\rangle_L \equiv a^\dagger_{\alpha_1}(L)\cdots a^\dagger_{\alpha_p}(L) b^{\dagger\beta_1}(L)\cdots b^{\dagger\beta_q}(L)|0\rangle_L$$

[7] We will analyze the consequences of $E^2_L(n, i) = E^2_R(n + i, i)$ in the next section.

and

$$\hat{R}^{\delta_1\delta_2\cdots\delta_p}_{\gamma_1\gamma_2\cdots\gamma_q}|0\rangle_R \equiv a^\dagger_{\gamma_1}(R)\cdots a^\dagger_{\gamma_q}(R)b^{\dagger\delta_1}(R)\cdots b^{\dagger\delta_p}(R)|0\rangle_R$$

are the $SU_L(3)\otimes SU_R(3)\otimes U(1)\otimes U(1)$ flux creation operators on the left and right ends of every link respectively. We have used the $U(1)\otimes U(1)$ Gauss law constraints (33) in (34) with $n_L = m_R = p$ and $m_L = n_R = q$. Note that unlike *SU*(2) flux creation operators (15) which were *SU*(2) irreducible, the flux operators in (34) are *SU*(3) reducible (see equation (38)). In this section we show that this is the reason why, unlike the *SU*(2) case (16), the *SU*(3) gauge theory Hilbert space $\mathcal{H}_g$ is contained in $\mathcal{H}_p$:

$$\mathcal{H}_g \subset \mathcal{H}_p. \tag{35}$$

Therefore, we now need projection operators to go from $\mathcal{H}_p$ to $\mathcal{H}_g$ (appendix A). This makes *SU*(3) prepotential analysis slightly more involved than *SU*(2) (see section 4.3). To appreciate this problem, we start with the following *SU*(3) gauge invariant state as an example:

$$|\rho_L,\rho_R\rangle \equiv (a^\dagger(L)\cdot b^\dagger(L))^{\rho_L}(a^\dagger(R)\cdot b^\dagger(R))^{\rho_R}|0\rangle. \tag{36}$$

The states (36) are also invariant under $U(1)\otimes U(1)$ gauge transformations (32) if $\rho_L = \rho_R = \rho$ with $\rho = 0,1,2,\ldots,\infty$. The state (36) with $\rho = 1$ is shown in figure 5. The gauge invariant states (36) are linear combinations of states in (34):

$$|\rho\rangle \equiv |\rho_L=\rho,\rho_R=\rho\rangle = \sum_{\vec{\alpha}} \left|{}^{\alpha_1\alpha_2\cdots\alpha_\rho}_{\alpha_1\alpha_2\cdots\alpha_\rho}\right\rangle_L \otimes \sum_{\vec{\beta}} \left|{}^{\beta_1\beta_2\cdots\beta_\rho}_{\beta_1\beta_2\cdots\beta_\rho}\right\rangle_R. \tag{37}$$

However, the only gauge invariant states in pure lattice gauge theories are the Wilson loop states residing around the plaquettes and *not on the links as* $\mathrm{Tr}(UU^\dagger) = \mathrm{Tr}(U^\dagger U) = 3$ on every link. In other words, the infinite towers of gauge invariant states (36) on different links do not exist in the lattice gauge theory. In fact, this issue of 'non-gauge theory states' in $\mathcal{H}_p$ is related to the well-known multiplicity problem in the direct products of *SU*(3). Note that the basis (34) in $\mathcal{H}_p$ is obtained by taking two direct products. The states $\hat{L}^{\beta_1\beta_2\cdots\beta_q}_{\alpha_1\alpha_2\cdots\alpha_p}|0\rangle_L$ and $\hat{R}^{\delta_1\delta_2\cdots\delta_p}_{\gamma_1\gamma_2\cdots\gamma_q}|0\rangle_R$ are individually direct products of quark and anti-quark irreducible representations: $(n_L = p, 0)_L \otimes (0, m_L = q)_L$ and $(n_R = q, 0)_R \otimes (0, m_R = p)_R$ respectively. Therefore, they can be further reduced using the *SU*(3) Clebsch Gordan series into irreps. of $SU_L(3)$ and $SU_R(3)$ respectively:

$$\begin{aligned}
(n_L = p, 0)_L \otimes (0, m_L = q)_L &= \sum_{\rho(L)=0}^{\min(p,q)} \oplus \underbrace{(p-\rho(L), q-\rho(L))_L}_{\mathcal{H}^L_p(p-\rho(L),q-\rho(L),\rho(L))},\\
(n_R = q, 0)_R \otimes (0, m_R = p)_R &= \sum_{\rho(R)=0}^{\min(p,q)} \oplus \underbrace{(q-\rho(R), p-\rho(R))_R}_{\mathcal{H}^R_p(q-\rho(R),p-\rho(R),\rho(R))}.
\end{aligned} \tag{38}$$

The multiplicities[8] occurring in such direct product representations have been extensively studied and classified in [21]. Following [21], we have defined $\mathcal{H}^l_p(p-\rho(l), q-\rho(l), \rho(l)), l = L, R$, mutually orthogonal Hilbert spaces as these Hilbert spaces are in different irreducible representations of $SU_l(3)$. As shown in appendix B, the *SU*(3) electric field constraints $E^2_L(n,i) = E^2_R(n+i,i)$ along with the $U(1)\otimes U(1)$ Gauss law constraints (33) on links imply

$$\rho(n,i;L) = \rho(n+i,i;R)$$

[8] Under *SU*(3) gauge transformations, the vectors in $\mathcal{H}^L_p(p,q,\rho)\otimes\mathcal{H}^R_p(q,p,\rho)$ in (39) transform as $(p,q)_L\otimes(q,p)_R$ irreducible representation of $SU_L(3)\otimes SU_R(3)$ independent of the value of ρ (=0, 1, ..., ∞) leading to infinite multiplicity for each state, while the gauge theory Hilbert space $\mathcal{H}_g$ contains each of these representations only once (see (46)).

in the $SU(3)$ Clebsch Gordan series (38) on every link. Therefore, the prepotential Hilbert space can be classified as

$$\mathcal{H}_p = \prod_{\otimes \text{link}} \{\mathcal{H}_p\}_{\text{link}} = \prod_{\otimes \text{link}} \left\{ \sum_{\rho=0}^{\infty} \sum_{p,q=0}^{\infty} \left(\mathcal{H}_p^L(p,q,\rho) \otimes \mathcal{H}_p^R(q,p,\rho)\right) \right\}_{\text{link}}$$
$$\equiv \prod_{\otimes \text{link}} \left\{ \sum_{\rho=0}^{\infty} \mathcal{H}_p(\rho) \right\}_{\text{link}}. \tag{39}$$

In order to identify the gauge theory Hilbert space $\mathcal{H}_g$ in (39), we define the following three color neutral operators on each side l of every link:

$$k_-(l) \equiv a(l) \cdot b(l), \qquad k_+(l) \equiv a^\dagger(l) \cdot b^\dagger(l),$$
$$k_0(l) \equiv \tfrac{1}{2}(a^\dagger(l) \cdot a(l) + b^\dagger(l) \cdot b(l) + 3), \qquad l \equiv L, R. \tag{40}$$

As usual, we have suppressed the link indices (n, i) in (40). These $SU(3)$ color neutral operators satisfy the $Sp(2, R)$ algebra on both sides of the link:

$$[k_0(l), k_\pm(l')] = \pm\delta_{l,l'} k_\pm(l), \qquad [k_-(l), k_+(l')] = 2\delta_{l,l'} k_0(l), \quad l, l' \equiv L, R. \tag{41}$$

Further, as these $Sp(2, R)$ generators are invariant under $SU(3)$ transformations, they commute with the color electric fields. In other words,

$$[Sp_L(2, R) \otimes Sp_R(2, R), SU_L(3) \otimes SU_R(3)] = 0. \tag{42}$$

Therefore, the Hilbert space of $SU(3)$ lattice gauge theory can be completely and uniquely labeled by $SU_L(3) \otimes Sp_L(2, R) \otimes SU_R(3) \otimes Sp_R(2, R)$ quantum numbers on every link. The irreducible representations of $Sp(2, R)$ are characterized by $|k, \rho\rangle$, where k and ρ represent the $Sp(2, R)$ 'spin' and 'magnetic' quantum numbers respectively. For the direct product (38) we get [21]: $k(L) = k(R) = \frac{1}{2}(p + q + 3)$. Further $\rho(L) = \rho(R)$ appearing in (38) are the 'magnetic quantum numbers' of $Sp_L(2, R) \otimes Sp_R(2, R)$. The raising (lowering) $K_+(K_-)$ operators increase (decrease) the $Sp(2, R)$ magnetic fluxes [21]:

$$\left|\mathcal{H}_p^L(p, q, \rho \pm 1)\right\rangle = k_\pm(L)\left|\mathcal{H}_p^L(p, q, \rho)\right\rangle,$$
$$\left|\mathcal{H}_p^R(q, p, \rho \pm 1)\right\rangle = k_\pm(R)\left|\mathcal{H}_p^R(q, p, \rho)\right\rangle, \tag{43}$$

where $\left|\mathcal{H}_p^l(p, q, \rho)\right\rangle$ denotes an arbitrary vector in $\mathcal{H}_p^l(p, q, \rho)$ with $l = L/R$. In particular, the $\rho = 0$ Hilbert space without any '$Sp(2, R)$ magnetic' flux in (38) is annihilated by k_-:

$$k_-(L)\left|\mathcal{H}_p^L(p, q, \rho = 0)\right\rangle = 0, \qquad k_-(R)\left|\mathcal{H}_p^R(q, p, \rho = 0)\right\rangle = 0. \tag{44}$$

Equations (43) show that the 'spurious gauge invariant' states in (36) are the vectors of one-dimensional mutually orthogonal $SU(3)$ invariant Hilbert spaces $\mathcal{H}_p^L(0, 0, \rho) \otimes \mathcal{H}_p^R(0, 0, \rho)$ with $\rho = 1, \ldots, \infty$. The strong coupling vacuum is the $\rho = 0$ vacuum.

4.3. The SU(3) gauge theory Hilbert space $\mathcal{H}_g$

The various flux states in gauge theory Hilbert space $\mathcal{H}_g$ are created by the link matrices $U^\alpha{}_\beta$ acting on the strong coupling vacuum as in (10). Therefore, in order to identify $\mathcal{H}_g$ in $\mathcal{H}_p$ with $Sp(2, R)$ structure (39), we now analyze the $Sp(2, R)$ properties of the link operators in this section. We note that the link matrix $U^\alpha{}_\beta$ cannot change the $Sp(2, R)$ magnetic quantum number ρ. As shown at the bottom of figure 5, $k_+(L) = a^\dagger(L) \cdot b^\dagger(L)$ and $k_+(R) = a^\dagger(R) \cdot b^\dagger(R)$ correspond to three Young tableau boxes in a vertical column ($SU(3)$ singlets) on the left and right sides of the links respectively. On the other hand, in terms of the link operators, this left and right anti-symmetrization on a link corresponds to $\frac{1}{3!}\epsilon_{\alpha_1\alpha_2\alpha_3}\epsilon^{\beta_1\beta_2\beta_3}U^{\alpha_1}{}_{\beta_1}U^{\alpha_2}{}_{\beta_2}U^{\alpha_3}{}_{\beta_3} = \det U \equiv 1$ or $\mathrm{Tr}(UU^\dagger) = 3$. Therefore, the states in $\mathcal{H}_g$, obtained by applying link operators on the strong coupling vacuum with $\rho = 0$ ($k_-(l)|0\rangle_l = 0, l = L, R$), will also carry $\rho = 0$ quantum numbers. In other words, they too will be annihilated by $k_-(l)$:

$$k_-(L)\big(U^{\alpha_1}{}_{\beta_1}U^{\alpha_2}{}_{\beta_2}\cdots U^{\alpha_r}{}_{\beta_r}\big)|0\rangle = 0, \qquad k_-(R)\big(U^{\alpha_1}{}_{\beta_1}U^{\alpha_2}{}_{\beta_2}\cdots U^{\alpha_r}{}_{\beta_r}\big)|0\rangle = 0. \tag{45}$$

Therefore, going back to the classification of $\mathcal{H}_p$ in (39), we identify

$$\mathcal{H}_g \equiv \prod_{\otimes \text{link}} \{\mathcal{H}_p(\rho = 0)\}_{\text{link}} \equiv \mathcal{H}_p^0 \tag{46}$$

like in the case of $SU(2)$ lattice gauge theory. In (46) $\mathcal{H}_p^0$ denotes $\rho = 0$ subspace of $\mathcal{H}_p$. Thus, the kernel of $(k_-(L)k_-(R))$ in $\mathcal{H}_p$ is the $SU(3)$ gauge theory Hilbert space $\mathcal{H}_g$. Further, (45) implies

$$\big[k_-(L), U^\alpha{}_\beta\big] \simeq 0, \qquad \big[k_-(R), U^\alpha{}_\beta\big] \simeq 0. \tag{47}$$

In other words, $k_-(L)$ and $k_-(R)$ weakly commute with the link operators of $SU(3)$ lattice gauge theory[9]. The symbol $\simeq$ in (47) implies that the commutators are zero only when they are applied on the vectors belonging to the gauge theory Hilbert space $\mathcal{H}_g$. We would now like to write the link operators in terms of $SU(3)$ prepotential operators which create $SU(3)$ fluxes only in the gauge theory Hilbert space $\mathcal{H}_g$. This is done in the next section.

4.4. *SU(3) irreducible prepotential operators*

In this section, we construct the $SU(3)$ irreducible prepotential operators from the prepotential operators in (28) such that they directly create $SU(3)$ irreducible fluxes exactly like in the $SU(2)$ case (15). This construction with all its group theoretical details is given in [22]. We define the $SU(3)$ irreducible prepotential operators from prepotential operators such that

(1) they have exactly the same $SU(3) \otimes U(1) \otimes U(1)$ quantum numbers,
(2) they commute with the $Sp(2, R)$ destruction operator k_-.

As a result, acting on the strong coupling vacuum they directly create the gauge theory Hilbert space $\mathcal{H}_g$ completely bypassing the problem of spurious states like (36) in $\mathcal{H}_p$. We define $SU(3)$ irreducible prepotentials [22] as

$$\begin{aligned} A^\dagger_\alpha(L) &= a^\dagger_\alpha(L) - F_L k_+(L) b_\alpha(L), & A^\dagger_\alpha(R) &= a^\dagger_\alpha(R) - F_R k_+(R) b_\alpha(R), \\ B^{\dagger\alpha}(L) &= b^{\dagger\alpha}(L) - F_L k_+(L) a^\alpha(L), & B^{\dagger\alpha}(R) &= b^{\dagger\alpha}(R) - F_L k_+(R) a^\alpha(R). \end{aligned} \tag{48}$$

In (48), the factors F_L and F_R are given by

$$F_L = \frac{1}{N(L) + M(L) + 1}, \qquad F_R = \frac{1}{N(R) + M(R) + 1}.$$

These factors are chosen so that [22]

$$\big[k_-(l), A^\dagger_\alpha(l)\big] \simeq 0; \qquad [k_-(l), B^{\dagger\alpha}(l)] \simeq 0. \tag{49}$$

It is easy to check that the irreducible Schwinger boson creation operators commute among themselves:

$$\big[A^\dagger_\alpha(l), A^\dagger_\beta(l')\big] = 0, \qquad [B^{\dagger\alpha}(l), B^{\dagger\beta}(l')] = 0, \qquad \big[A^\dagger_\alpha(l), B^{\dagger\beta}(l')\big] = 0. \tag{50}$$

[9] Note that all the electric fields strongly commute with the $Sp(2, R)$ generators (42).

The other commutation relations acting on the $SU(3)$ irreps. are [22]

$$\begin{aligned}
\left[A^{\alpha}(l), A^{\dagger}_{\beta}(l')\right] &\simeq \delta_{ll'}\left(\delta^{\beta}_{\alpha} - \frac{1}{N(l)+M(l)+2}B^{\dagger\alpha}B_{\beta}\right)\\
[A^{\alpha}(l), B^{\dagger\beta}(l')] &\simeq -\delta_{ll'}\frac{1}{N(l)+M(l)+2}B^{\dagger\alpha}A^{\beta}\\
[B_{\alpha}(l), B^{\dagger\beta}(l')] &\simeq \delta_{ll'}\left(\delta^{\alpha}_{\beta} - \frac{1}{N(l)+M(l)+2}A^{\dagger}_{\alpha}A^{\beta}\right).
\end{aligned} \tag{51}$$

By construction, $A^{\dagger}_{\alpha}(l)$ and $B^{\dagger\alpha}(l)$ transform exactly like $a^{\dagger}_{\alpha}(l)$ and $b^{\dagger\alpha}(l)$, $l = L, R$, under $SU(3) \otimes U(1) \otimes U(1)$ and retain the same quantum numbers. Therefore, we can now define

$$\left|{}^{\beta_1\beta_2\cdots\beta_q}_{\alpha_1\alpha_2\cdots\alpha_p}\right\rangle^{0}_{L} \otimes \left|{}^{\delta_1\delta_2\cdots\delta_p}_{\gamma_1\gamma_2\cdots\gamma_q}\right\rangle^{0}_{R} \equiv \hat{\mathcal{L}}^{\beta_1\beta_2\cdots\beta_q}_{\alpha_1\alpha_2\cdots\alpha_p}|0\rangle_L \otimes \hat{\mathcal{R}}^{\delta_1\delta_2\cdots\delta_p}_{\gamma_1\gamma_2\cdots\gamma_q}|0\rangle_R. \tag{52}$$

In (52), the additional $Sp(2, R)$ quantum numbers $\rho_L = \rho_R = 0$ are put as superscript 0. The operators $\mathcal{L}$ and $\mathcal{R}$ are defined by replacing $SU(3)$ prepotentials in L and R in (34) by the corresponding $SU(3)$ irreducible prepotentials in (48), i.e.,

$$\hat{\mathcal{L}}^{\beta_1\beta_2\cdots\beta_q}_{\alpha_1\alpha_2\cdots\alpha_p}|0\rangle_L \equiv A^{\dagger}_{\alpha_1}(L)\cdots A^{\dagger}_{\alpha_p}(L)B^{\dagger\beta_1}(L)\cdots B^{\dagger\beta_q}(L)|0\rangle_L,$$

and

$$\hat{\mathcal{R}}^{\delta_1\delta_2\cdots\delta_p}_{\gamma_1\gamma_2\cdots\gamma_q}|0\rangle_R \equiv A^{\dagger}_{\gamma_1}(R)\cdots A^{\dagger}_{\gamma_q}(R)B^{\dagger\delta_1}(R)\cdots B^{\dagger\delta_p}(R)|0\rangle_R.$$

Note that in terms of $SU(3)$ irreducible prepotentials, the 'spurious gauge invariant states' like in (36) or (37) do not exist as

$$A^{\dagger}(L)\cdot B^{\dagger}(L)|0\rangle_L \equiv 0, \qquad A^{\dagger}(R)\cdot B^{\dagger}(R)|0\rangle_R \equiv 0. \tag{53}$$

In other words, the operators $\mathcal{L}$ and $\mathcal{R}$ in (52) are $SU(3)$ irreducible unlike the L and R operators in (34) which are reducible according to (38). In appendix A we show that $\mathcal{L}$ and $\mathcal{R}$ are related to L and R by projection operators (A.9). In fact, these $SU(3)$ flux creation operators $\mathcal{L}$ and $\mathcal{R}$ are the $SU(3)$ analogs of the $SU(2)$ flux creation operators $\mathcal{L}$ and $\mathcal{R}$ in (15) as both create irreducible fluxes. Further, like in the $SU(2)$ case, they bypass the problem of symmetrization and anti-symmetrization associated with the link operators. This is because $\hat{\mathcal{L}}$ and $\hat{\mathcal{R}}$ in (52) are defined in terms of $SU(3)$ irreducible prepotential operators which have all the symmetries of $SU(3)$ Young tableaus inbuilt [22]. In other words, the role played by $SU(2)$ prepotentials in $SU(2)$ lattice gauge theory is exactly equivalent to the role played by $SU(3)$ irreducible prepotentials in $SU(3)$ lattice gauge theory.

4.5. SU(3) link operators

The $SU(3)$ link operator must create 3 and 3* fluxes at the left and right ends of the link and should satisfy the $U(1) \otimes U(1)$ Gauss law constraints (33). These requirements are similar to the $SU(2)$ case discussed in section 3.2. The new $SU(3)$ requirement of $Sp(2, R)$ constraint (47) has been solved by defining $SU(3)$ irreducible prepotential operators in the previous section. Noting that by construction, $A^{\dagger}_{\alpha}(l)$ and $B^{\dagger\alpha}(l)$ transform exactly like $a^{\dagger}_{\alpha}(l)$ and $b^{\dagger\alpha}(l)$, $l = L, R$, and the general structure of the link operator is

$$U^{\alpha}{}_{\beta} = B^{\dagger\alpha}(L)\eta A^{\dagger}_{\beta}(R) + A^{\alpha}(L)\theta B_{\beta}(R) + (B(L)\wedge A^{\dagger}(L))^{\alpha}\delta(A(R)\wedge B^{\dagger}(R))_{\beta}. \tag{54}$$

In (54), η, θ and δ are the $SU(3)$ invariants and therefore can only depend on the number operators. These will be fixed later in this section. The link operator constructed in (54) has all the required group theoretical properties.

- Under $SU(3)$ transformations $U(n,i)^{\alpha}{}_{\beta} \to (\Lambda_L)^{\alpha}{}_{\gamma}U(n,i)^{\gamma}{}_{\delta}(\Lambda_R{}^{\dagger})^{\delta}{}_{\beta}$.

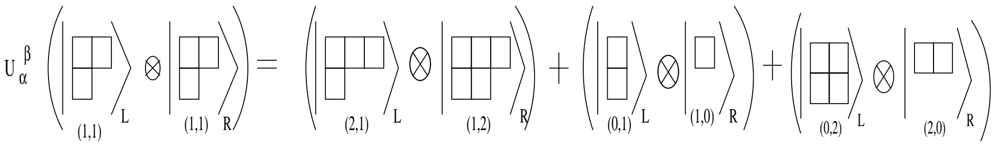

Figure 6. The Young tableau interpretation of the $SU(3)$ link operator U in terms of the prepotential operators (54) acting on a state with $n_L = m_R \equiv p = 1$ and $m_L = n_R \equiv q = 1$. The three terms in (54) or (55) correspond to the three sets of (mutually conjugate) Young tableaus on the right-hand side of this figure respectively. This is $SU(3)$ generalization of figure 2 for $SU(2)$.

- It is invariant under $U(1) \otimes U(1)$ Abelian gauge transformations.
- It creates and destroys fluxes in $\mathcal{H}_p^0$ in (46). It is easy to check that the link operator $U^{\alpha}{}_{\beta}$ in (54) satisfy (47).
- Acting on a link state in $(p, q)_L$ and $(q, p)_R$ representations of $SU(3)_L \times SU(3)_R$:

$$\begin{aligned} U^{\alpha}{}_{\beta}|p,q\rangle_L \otimes |q,p\rangle_R = {} & C_1{}^{\alpha}{}_{\beta}|p+1,q\rangle_L \otimes |q,p+1\rangle_R \\ & + C_2{}^{\alpha}{}_{\beta}|p,q-1\rangle_L \otimes |q-1,p\rangle_R \\ & + C_3{}^{\alpha}{}_{\beta}|p-1,q+1\rangle_L \otimes |q+1,p-1\rangle_R, \end{aligned} \tag{55}$$

where C_1, C_2 and C_3 are the $SU(3)$ Clebsch Gordan coefficients. The three terms in (54) correspond to the three terms in (55) respectively. In figure 6, we illustrate (54) and (55) in terms of $SU(3)$ Young tableau diagrams.

Like in the $SU(2)$ case, it is convenient to define left and right link operators as

$$U = \underbrace{\begin{pmatrix} B^{\dagger 1}(L)\eta_L & A^1(L)\theta_L & (B(L)\wedge A^{\dagger}(L))^1\delta_L \\ B^{\dagger 2}(L)\eta_L & A^2(L)\theta_L & (B(L)\wedge A^{\dagger}(L))^2\delta_L \\ B^{\dagger 3}(L)\eta_L & A^3(L)\theta_L & (B(L)\wedge A^{\dagger}(L))^3\delta_L \end{pmatrix}}_{U_L} \underbrace{\begin{pmatrix} A^1(R)\bar{\eta}_R & B^{1\dagger}(R)\bar{\theta}_R & (B(R)\wedge A^{\dagger}(R))^1\bar{\delta}_R \\ A^2(R)\bar{\eta}_R & B^{2\dagger}(R)\bar{\theta}_R & (B(R)\wedge A^{\dagger}(R))^2\bar{\delta}_R \\ A^3(R)\bar{\eta}_R & B^{3\dagger}(R)\bar{\theta}_R & (B(R)\wedge A^{\dagger}(R))^3\bar{\delta}_R \end{pmatrix}^{\dagger}}_{U_R} \tag{56}$$

where $\eta_L, \theta_L, \delta_L$ and $\bar{\eta}_R, \bar{\theta}_R, \bar{\delta}_R$ are the left and right invariants constructed out of the number operators. From (54),

$$\eta = \eta_L\eta_R, \qquad \theta = \theta_L\theta_R, \qquad \delta = \delta_L\delta_R. \tag{57}$$

From (56),

$$U_L^{\dagger}U_L = \begin{pmatrix} \bar{\eta}_L(B\cdot B^{\dagger})\eta_L & \bar{\eta}_L\underbrace{(B\cdot A)}_{\simeq 0}\theta_L & \bar{\eta}_L\underbrace{(B\cdot(B\wedge A^{\dagger}))}_{\equiv 0}\delta_L \\ \bar{\theta}_L\underbrace{(A^{\dagger}\cdot B^{\dagger})}_{\simeq 0}\eta_L & \bar{\theta}_L(A^{\dagger}\cdot A)\theta_L & \bar{\theta}_L\underbrace{(A^{\dagger}\cdot(B\wedge A^{\dagger}))}_{\equiv 0}\delta_L \\ \bar{\delta}_L\underbrace{((B^{\dagger}\wedge A)\cdot B^{\dagger})}_{\equiv 0}\eta_L & \bar{\delta}_L\underbrace{((B^{\dagger}\wedge A)\cdot A)}_{\equiv 0}\theta_L & \bar{\delta}_L((A\wedge B^{\dagger})\cdot(B\wedge A^{\dagger}))\delta_L \end{pmatrix}. \tag{58}$$

Similarly,

$$U_RU_R^{\dagger} = \begin{pmatrix} \eta_R(A^{\dagger}\cdot A)\bar{\eta}_R & \eta_R\underbrace{(A^{\dagger}\cdot B^{\dagger})}_{\simeq 0}\bar{\theta}_R & \eta_R\underbrace{(A^{\dagger}\cdot(B\wedge A^{\dagger}))}_{\equiv 0}\bar{\delta}_R \\ \theta_R\underbrace{(B\cdot A)}_{\simeq 0}\bar{\eta}_R & \theta_R(B\cdot B^{\dagger})\bar{\theta}_R & \theta_R\underbrace{(B\cdot(B\wedge A^{\dagger}))}_{\equiv 0}\bar{\delta}_R \\ \delta_R\underbrace{((B^{\dagger}\wedge A)\cdot A)}_{\equiv 0}\bar{\eta}_R & \delta_R\underbrace{((B^{\dagger}\wedge A)\cdot B^{\dagger})}_{\equiv 0}\bar{\theta}_R & \delta_R((A\wedge B^{\dagger})\cdot(B\wedge A^{\dagger}))\bar{\delta}_R \end{pmatrix}. \tag{59}$$

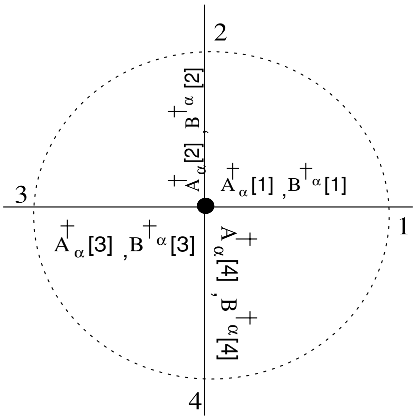

Figure 7. The $SU(3)$ prepotentials associated with a lattice site n in $d = 2$. This is $SU(3)$ generalization of figure (3) for $SU(2)$.

In (58) and (59), we have suppressed the L/R indices from the prepotential operators $(A, A^\dagger)$ and $(B, B^\dagger)$. Demanding $U_L^\dagger U_L = 1$ and $U_R U_R^\dagger = 1$, we get

$$\begin{aligned}
\eta_L &= \frac{1}{\sqrt{B(L)\cdot B^\dagger(L)}}, & \theta_L &= \frac{1}{\sqrt{A^\dagger(L)\cdot A(L)}},\\
\delta_L &= \frac{1}{\sqrt{(A(L)\wedge B^\dagger(L))\cdot(B(L)\wedge A^\dagger(L))}}, & \eta_R &= \frac{1}{\sqrt{A^\dagger(R)\cdot A(R)}},\\
\theta_R &= \frac{1}{\sqrt{B(R)\cdot B^\dagger(R)}}, & \delta_R &= \frac{1}{\sqrt{(A(R)\wedge B^\dagger(R))\cdot(B(R)\wedge A^\dagger(R))}}.
\end{aligned} \tag{60}$$

The link operators in (54) with (57) and (60) satisfy $UU^\dagger = U^\dagger U = 1$. Having written the link operators in terms of the $SU(3)$ irreducible prepotentials, we now cast the left and right electric fields (28) in terms of $A(l), A^\dagger(l), B(l), B^\dagger(l)$ with $l = L, R$. Using the very special structures of the $SU(3)$ irreducible prepotentials in (48), it is easy to check that

$$\begin{aligned}
E_L^{\mathrm{a}} &= \left(a^\dagger(L)\frac{\lambda^{\mathrm{a}}}{2}a(L) - b(L)\frac{\lambda^{\mathrm{a}}}{2}b^\dagger(L)\right) \simeq \left(A^\dagger(L)\frac{\lambda^{\mathrm{a}}}{2}A(L) - B(L)\frac{\lambda^{\mathrm{a}}}{2}B^\dagger(L)\right)\\
E_R^{\mathrm{a}} &= \left(a^\dagger(R)\frac{\lambda^{\mathrm{a}}}{2}a(R) - b(R)\frac{\lambda^{\mathrm{a}}}{2}b^\dagger(R)\right) \simeq \left(A^\dagger(R)\frac{\lambda^{\mathrm{a}}}{2}A(R) - B(R)\frac{\lambda^{\mathrm{a}}}{2}B^\dagger(R)\right).
\end{aligned} \tag{61}$$

In (61), we have made use of the identities: $a(L)\cdot b(L) \equiv k_-(L) \simeq 0$ and $a(R)\cdot b(R) \equiv k_-(R) \simeq 0$ on every link of the lattice. In fact results (61) were expected because $\left(a_\alpha^\dagger, b^{\dagger\beta}\right)$ and $\left(A_\alpha^\dagger, B^{\dagger\beta}\right)$ have exactly the same $SU(3)\otimes U(1)\otimes U(1)$ transformation properties.

4.6. SU(3) gauge invariant states and Mandelstam constraints

In this section we construct all possible $SU(3)$ gauge invariant states at a given lattice site using the prepotential approach. We also discuss the Mandelstam constraints which relate these gauge invariant states. The additional $U(1)\otimes U(1)$ Gauss law (33) can be satisfied by drawing the Abelian flux lines along the links as is done in figure 8. As shown in figure 7, every lattice site in $2d$ space dimension is associated with $2d$ pairs of quark–anti-quark prepotentials $\left(A_\alpha^\dagger, B^{\dagger\alpha}\right)$. Under a gauge transformation at site n, all these $2d$ quark

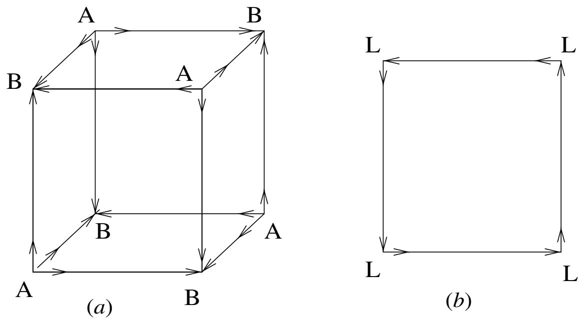

Figure 8. Graphical representation of the three possible $SU(3)$ gauge invariant L, A, B types of vertices. Two simple $SU(3) \otimes U(1) \otimes U(1)$ gauge invariant loop states are also shown. The arrows represent the directions of the Abelian (non-Abelian) fluxes on the links (sites).

(anti-quark) prepotentials transform together as a triplet (anti-triplet). Therefore, the fundamental $SU(3)$ gauge invariant creation operator vertices at a lattice site n are

$$L_{[ij]} \equiv A^{\dagger}[i] \cdot B^{\dagger}[j], \qquad i \neq j, \tag{62}$$

$$A_{[i_1,i_2,i_3]} = \epsilon^{\alpha_1\alpha_2\alpha_3} A^{\dagger}_{\alpha}[i_1] A^{\dagger}_{\alpha_2}[i_2] A^{\dagger}_{\alpha_3}[i_3], \tag{63}$$

$$B_{[j_1,j_2,j_3]} = \epsilon_{\beta_1\beta_2\beta_3} B^{\dagger\beta_1}[j_1] B^{\dagger\beta_2}[j_2] B^{\dagger\beta_3}[j_3]. \tag{64}$$

These vertices are shown in figure 8. We have taken $i \neq j$ in (62) because $L_{ii} = A^{\dagger}[i] \cdot B^{\dagger}[i] \simeq 0, i, j = 1, 2, \ldots, 2d$, according to (47). Also, $A_{[i_1,i_2,i_3]}$ and $B_{[j_1,j_2,j_3]}$ are completely anti-symmetric in (i_1, i_2, i_3) and (j_1, j_2, j_3) indices respectively. The above

$$2(^{2d}C_2) + 2(^{2d}C_3) = \frac{2d(2d-1)(2d+1)}{3}$$

basic $SU(3)$ gauge invariant operators enable us to write the most general $SU(3)$ gauge invariant state at a given lattice site as

$$|\vec{l}_{[ij]}, \vec{p}_{[i_1i_2i_3]}, \vec{q}_{[j_1j_2j_3]}\rangle = \prod_{\substack{i,j=1\\ i\neq j}}^{2d} (L_{[ij]})^{l_{[ij]}} \prod_{[i_1i_2i_3]=1}^{^{2d}C_3} \left(A_{[i_1i_2i_3]}\right)^{p_{[i_1i_2i_3]}} \prod_{[j_1j_2j_3]=1}^{^{2d}C_3} \left(B_{[j_1j_2j_3]}\right)^{q_{[j_jj_2j_3]}} |0\rangle. \tag{65}$$

In (65), $\vec{l}_{[ij]}, \vec{p}_{[i_1i_2i_3]}, \vec{q}_{[j_1j_2j_3]}$ are $\frac{2d(2d-1)(2d+1)}{3}$ non-negative integers describing all possible $SU(3)$ gauge invariant states at a given lattice site. The various possible loop states set in pure $SU(3)$ lattice gauge theory are direct products of (65) at various lattice sites consistent with $U(1) \otimes U(1)$ Gauss law (33) along every link.

As in the loop formulation where various loop states are mutually related by Mandelstam constraints, not all states in (65) are linearly independent. In fact, in the present $SU(3)$ prepotential formulation (like in the $SU(2)$ case) the Mandelstam constraints become local and take very simple forms in terms of the $SU(3)$ gauge invariant vertices in (62), (63) and (64) at every lattice site n. We start with the simplest $SU(3)$ Mandelstam constraints:

$$A_{[i_1,i_2,i_3]} B_{[j_1,j_2,j_3]} \equiv \sum_{\{s_1,s_2,s_3\}\in S_3} (-1)^s L_{[i_1j_{s_1}]} L_{[i_2j_{s_2}]} L_{[i_3j_{s_3}]}. \tag{66}$$

In (66), S_3 denotes the permutation group of order 3, $\{s_1, s_2, s_3\}$ denote the 3! permutations of $\{1, 2, 3\}$ and s is the parity of permutation. In other words, the Mandelstam constraints

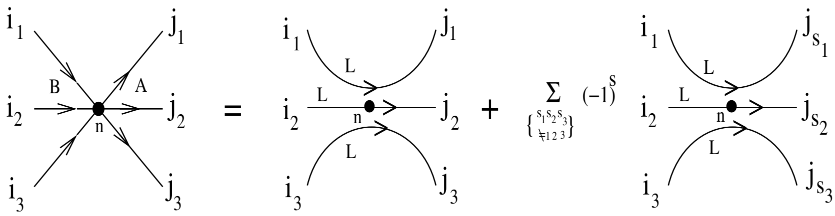

Figure 9. The graphical representation of local $SU(3)$ Mandelstam constraints (66) in terms of $SU(3)$ gauge invariant vertices A, B and L constructed out of the $SU(3)$ irreducible prepotential operators at a lattice site n. The A and B type of vertices at n annihilate each other to produce L type of vertices.

(66) state that the A and B type vertices annihilate each other in pairs to produce L type vertices. The constraints (66) are illustrated in figure 9. Therefore, the $SU(3)$ gauge invariant states of (L–A–B) type in (65) can always be written either as (L–A) type or as (L–B) type at each lattice site. It is interesting to analyze the Mandelstam constraints discussed in [4] in terms of $SU(3)$ prepotential operators. Following [4], we consider the set of r ($r > 3$) loops $C_1(n), C_2(n), \ldots, C_r(n)$ all based at lattice site n. These loops start from n in the directions $i_1, i_2, \ldots, i_r$ and come back to n from directions $j_1, j_2, \ldots, j_r$ respectively. Then the products of these Wilson loops satisfy

$$\sum_{\substack{\alpha_{i_1}\cdots\alpha_{i_r}\\ \beta_{j_1}\cdots\beta_{j_r}}} \epsilon_{\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_r}} \epsilon^{\beta_{j_1}\beta_{j_2}\cdots\beta_{j_r}} (W(C_1(n))^{\alpha_{j_1}}{}_{\beta_{i_1}} (W(C_2(n))^{\alpha_{j_2}}{}_{\beta_{i_2}} \cdots (W(C_r(n))^{\alpha_{j_r}}{}_{\beta_{i_r}} \equiv 0. \tag{67}$$

Using the identities

$$\epsilon_{\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_r}} \epsilon^{\beta_{j_1}\beta_{j_2}\cdots\beta_{j_r}} = \delta^{\beta_{j_1}}_{\alpha_{i_1}} \delta^{\beta_{j_2}}_{\alpha_{i_2}} \cdots \delta^{\beta_{j_r}}_{\alpha_{i_r}} - \delta^{\beta_{j_1}}_{\alpha_{i_2}} \delta^{\beta_{j_2}}_{\alpha_{i_i}} \cdots \delta^{\beta_{j_r}}_{\alpha_{i_r}} + \cdots$$

(68) can be written in terms of traces of Wilson loops [4]:

$$\mathrm{Tr}\, W(C_1)\mathrm{Tr}\, W(C_2)\cdots \mathrm{Tr}\, W(C_r) - \mathrm{Tr}\, W(C_1C_2)\mathrm{Tr}\, W(C_3)\mathrm{Tr}\, W(C_4)\cdots \mathrm{Tr}\, W(C_r) + \cdots = 0. \tag{68}$$

The Mandelstam constraints (68) in terms of the link operators represent highly non-local constraints as one can always choose the loops $C_1, C_2, \ldots, C_r$ to be as large as one wishes. However, in terms of the prepotentials the constraints (68) become local. All one has to do is to replace the Wilson loops in (68) by the prepotentials which are attached to their starting and end points, i.e.

$$W(C_s)^{\alpha_{i_s}}{}_{\beta_{i_s}} \to L^{\alpha_{j_s}}{}_{\beta_{i_s}} \equiv B^{\dagger\alpha_{j_s}} A^{\dagger}_{\beta_{i_s}}, \qquad s = 1, 2, \ldots, r. \tag{69}$$

Note that unlike the non-local Wilson loop $W(C_s)$, the operators $B^{\alpha_{j_s}}$ and $A^{\dagger}_{\beta_{i_s}}$ and hence $L^{\alpha_{j_s}}{}_{\beta_{i_s}}$ are completely defined at lattice site n. Noting that $\mathrm{Tr}\left(L^{\alpha_{j_s}}{}_{\beta_{i_s}}\right) = L[i_s j_s]$, the non-local Mandelstam constraints (68) acquire the following simple local form:

$$\sum_{\{s_1,s_2,\ldots,s_r\}\in S_r} (-1)^s L_{[i_1 j_{s_1}]} L_{[i_2 j_{s_2}]} \cdots L_{[i_3 j_{s_r}]} = 0 \tag{70}$$

and are illustrated in figure 10. *Note that all the unnecessary details such as shapes, sizes and lengths of the loops $C_1, C_2, \ldots, C_r$ in (68) disappear in the corresponding prepotential form (70).*

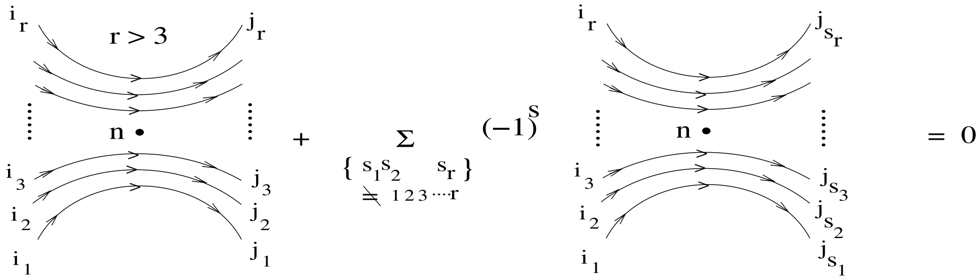

Figure 10. The graphical representation of local $SU(3)$ Mandelstam constraints (70) involving only L type of vertices at lattice site n.

4.6.1. The solutions. The Mandelstam constraints in their present local prepotential forms (66) and (70), instead of non-local form (68) in terms of link operators, are now accessible to explicit local solutions like in $SU(2)$ lattice gauge theory [8]. Note that they are still infinite in number at every lattice site. The solutions must be all possible mutually independent linear combinations of the states in (65) at a given lattice site. Following the techniques discussed in [10] in the context of duality transformations in lattice gauge theories, these linear combinations can be obtained by characterizing the resultant states at a site n by their complete $SU(3)$ quantum numbers with the net $SU(3)$ fluxes being zero. This will be $SU(3)$ analog of $SU(2)$ result (27). The quantum numbers needed to specify such states can be easily computed [10] as follows. In d dimension, there are $2d$ links emanating from a lattice site n. Each of these $2d$ directions is attached with $SU(3)$ operators $(A^\dagger[i], B^\dagger[i], i = 1, 2, \ldots, 2d)$ as shown in figure 7. Therefore, there are $2d$ Hilbert spaces associated with a lattice site and each can be characterized by its $SU(3)$ quantum numbers. In the standard language [24], the $SU(3)$ irreducible representations are completely specified by five quantum numbers: $|p, q, i^2, i_z, y\rangle$ where p and q are the eigenvalues of two $SU(3)$ Casimir operators and i, i_z, y are the $SU(3)$ 'magnetic' quantum numbers representing $SU(2)$ spin, its third component and hyper charge respectively. In the present language with constraints, each of the $2d$ directions (see figure 7) is associated with six harmonic oscillators $(A^\dagger[i], B^\dagger[i], i = 1, 2, \ldots, 2d)$ and therefore requires six occupation numbers to completely specify the basis. The constraints $k_-[i] \equiv a[i] \cdot b[i] \simeq 0$ reduce this to 5 in each direction. Therefore, $5 \times 2d = 10d$ quantum numbers are needed to specify a local Hilbert space basis completely at each lattice site. Not all these quantum numbers are independent as $2d$ of these are related to the previous sites by $U(1) \otimes U(1)$ Gauss law constraints (33). Therefore, we are left with $8d$ quantum numbers at every lattice site. Finally, the $SU(3)$ gauge invariance further implies eight constraints. Therefore, the net independent quantum numbers are $8(d-1)$ per lattice site. As expected, this is the the number of transverse degree of freedom of 8 $SU(3)$ gluons in d dimension at every lattice site. The Abelian $U(1) \times U(1)$ fluxes over the links will now glue these local $SU(3)$ invariant orthogonal basis at neighboring lattice sites according to their Gauss laws (33). This will give complete solutions of all the $SU(3)$ Mandelstam constraints like what was done in $SU(2)$ lattice gauge theory [8]. In fact, the addition of fluxes in $SU(3)$ lattice gauge theory has been discussed in [27]). These results combined with the results of this work should enable us to solve $SU(3)$ Mandelstam constraints completely in terms of vertex operators of section 4.6. This explicit construction of all the independent $SU(3)$ loop states and their dynamics along the line of [8] is in progress and will be reported elsewhere.

5. Summary and discussion

In this work we analyze $SU(3)$ lattice gauge theory in terms of the prepotential operators which under gauge transformations transform like fundamental matter fields. We constructed the $SU(3)$ irreducible prepotential operators which acting on strong coupling vacuum directly created the QCD fluxes around lattice sites. All $SU(3)$ gauge invariant vertices in terms of these QCD flux operators were constructed at every lattice site. These $SU(3)$ invariant vertices, in turn, enabled us to cast all $SU(3)$ Mandelstam constraints in their local forms. As mentioned in the text this is an essential step toward their complete solution. The complete solution of Mandelstam constraints, in turn, will allow us to write down $SU(3)$ lattice gauge theory completely and exactly in terms of minimum essential gauge invariant loop and string coordinates without any redundant loop/strings degrees of freedom. The prepotential operators also allow us to simplify lattice gauge theory Hamiltonian as given in (1). In particular, for the present $SU(3)$ case, one can simply replace the plaquette or magnetic term $\mathrm{Tr}U_{\mathrm{plaquette}}$ in (1) by a new plaquette interaction consisting of the 4 L type vertices at the four corners of every plaquette. Note that the new Hamiltonian constructed has exactly the same symmetries as (1) and therefore expected to be in the same universality class. The addition of matter field interactions in the prepotential formulation is trivial as matter and prepotential have similar $SU(3)$ gauge transformation properties. The difference lies in the Abelian $U(1) \otimes U(1)$ transformations under which matter fields remain invariant.

The results of this work can also be generalized to $SU(N)$ lattice gauge theory. We can use $SU(N)$ Schwinger bosons [26] or prepotentials to construct $SU(N)$ electric fields on lattice similar to (11) and (28). We need to elevate these prepotentials so that they have symmetries of $SU(N)$ Young tableaus inbuilt. As in section 4.6, the $SU(N)$ Mandelstam constraints will again be local and can be solved using the techniques discussed in this work. The work in this direction is in progress and will be reported elsewhere.

Acknowledgments

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Appendix A. The projection operators in $\mathcal{H}_p$

In this appendix we briefly discuss the construction of projection operators which project $\mathcal{H}_p$ to $\mathcal{H}_g$ on every link:

$$\mathcal{P}\{\mathcal{H}_p\}_{\mathrm{link}} = \{\mathcal{H}_p(\rho = 0)\}_{\mathrm{link}} = \{\mathcal{H}_g\}_{\mathrm{link}}. \tag{A.1}$$

The group theoretical details of this construction can be found in [22]. It is convenient to first break up $\{\mathcal{H}_p\}_{\mathrm{link}}$ into Hilbert spaces containing p (q) quarks and q (p) anti-quark prepotentials on the left (right):

$$\{\mathcal{H}_p\}_{\mathrm{link}} = \sum_{p,q=0}^{\infty} \oplus \{\mathcal{H}_p\}_{\mathrm{link}}(p, q). \tag{A.2}$$

These subspaces $\{\mathcal{H}_p\}_{\mathrm{link}}(p, q)$ are themselves direct products of left and right Hilbert spaces:

$$\{\mathcal{H}_p\}_{\mathrm{link}}(p, q) = \left\{\mathcal{H}_p^L\right\}_{\mathrm{link}}(p, q) \otimes \left\{\mathcal{H}_p^R\right\}_{\mathrm{link}}(q, p). \tag{A.3}$$

The basis vectors spanning $\left\{\mathcal{H}_p^l\right\}_{\mathrm{link}}(p, q)$, $l = L, R$, are given in terms of left and right flux creation operators:

$$\hat{L}^{\beta_1\beta_2\cdots\beta_q}_{\alpha_1\alpha_2\cdots\alpha_p}|0\rangle_L \equiv a^{\dagger}_{\alpha_1}(L)\cdots a^{\dagger}_{\alpha_p}(L) b^{\dagger\beta_1}(L)\cdots b^{\dagger\beta_q}(L)|0\rangle_L,$$

and

$$\hat{R}^{\delta_1\delta_2\cdots\delta_p}_{\gamma_1\gamma_2\cdots\gamma_q}|0\rangle_R \equiv a^{\dagger}_{\gamma_1}(R)\cdots a^{\dagger}_{\gamma_q}(R)b^{\dagger\delta_1}(R)\cdots b^{\dagger\delta_p}(R)|0\rangle_R$$

in (34). We now construct the projection operators $\mathcal{P}_l(p,q)$ in each of these subspaces with

$$\mathcal{P} = \sum_{p,q=0}^{\infty} \oplus \mathcal{P}_L(p,q) \otimes \mathcal{P}_R(q,p). \tag{A.4}$$

The left and right projection operators $\mathcal{P}_l(p,q), l = L, R$ are of the form [22]:

$$\begin{aligned} \mathcal{P}_L(p,q) &\equiv \sum_{r=0}^{\infty} g_r(p,q)\,(k_+(L))^r\,(k_-(L))^r \\ \mathcal{P}_R(q,p) &\equiv \sum_{r=0}^{\infty} h_r(q,p)(k_+(R))^r(k_-(R))^r. \end{aligned} \tag{A.5}$$

The unknown coefficients g and h in (A.5) are fixed by demanding the $Sp(2,R)$ constraints (44):

$$k_-(L)\mathcal{P}_L(p,q)\left\{\mathcal{H}^L_p\right\}_{\text{link}}(p,q) = 0, \qquad k_-(R)\mathcal{P}_R(q,p)\left\{\mathcal{H}^R_p\right\}_{\text{link}}(q,p) = 0. \tag{A.6}$$

The solutions of equations (A.6) are [22]

$$g_r(p,q) = h_r(q,p) = \frac{(-1)^r}{r!}\frac{(p+q+1-r)!}{(p+q+1)!}, \tag{A.7}$$

leading to

$$\begin{aligned} \mathcal{P}_L(p,q) &= \frac{1}{(p+q+1)!}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}(p+q+1-r)!\,(k_+(L))^r\,(k_-(L))^r, \\ \mathcal{P}_R(q,p) &= \frac{1}{(p+q+1)!}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}(p+q+1-r)!(k_+(R))^r(k_-(R))^r. \end{aligned} \tag{A.8}$$

Note that the $SU(3)$ irreducible prepotentials in (48) already commute with the $Sp(2,R)$ constraints (49) and therefore acting on the strong coupling vacuum directly generate the gauge theory Hilbert space $\mathcal{H}_g$. In other words,

$$\hat{\mathcal{L}}^{\beta_1\beta_2\cdots\beta_q}_{\alpha_1\alpha_2\cdots\alpha_p}|0\rangle_L = \mathcal{P}_L(p,q)\hat{L}^{\beta_1\beta_2\cdots\beta_q}_{\alpha_1\alpha_2\cdots\alpha_p}|0\rangle_L, \qquad \hat{\mathcal{R}}^{\delta_1\delta_2\cdots\delta_p}_{\gamma_1\gamma_2\cdots\gamma_q}|0\rangle_R. = \mathcal{P}_R(q,p)\hat{R}^{\delta_1\delta_2\cdots\delta_p}_{\gamma_1\gamma_2\cdots\gamma_q}|0\rangle_R \tag{A.9}$$

are the relations among the $SU(3)$ reducible and irreducible flux operators on the left and right sides of every link.

Appendix B. The electric field constraints

Using the λ matrix identity: $\sum_{a=1}^{8}\left(\frac{\lambda^a}{2}\right)^{\alpha}_{\beta}\left(\frac{\lambda^a}{2}\right)^{\gamma}_{\sigma} = \frac{1}{2}\delta^{\alpha}_{\sigma}\delta^{\gamma}_{\beta} - \frac{1}{6}\delta^{\alpha}_{\beta}\delta^{\gamma}_{\sigma}$, the squares of left and right electric fields can be written as

$$\begin{aligned} \sum_{a=1}^{8}E^a_L(n,i)E^a_L(n,i) &= \hat{N}(L)\left(\frac{\hat{N}(L)}{3}+1\right) \\ &\quad + \hat{M}(L)\left(\frac{\hat{M}(L)}{3}+1\right) - k_+(L)k_-(L) + \frac{1}{3}\hat{N}(L)\hat{M}(L) \\ \sum_{a=1}^{8}E^a_R(n,i)E^a_R(n,i) &= \hat{N}(R)\left(\frac{\hat{N}(R)}{3}+1\right) \\ &\quad + \hat{M}(R)\left(\frac{\hat{M}(R)}{3}+1\right) - k_+(R)k_-(R) + \frac{1}{3}\hat{N}(R)\hat{M}(R). \end{aligned}$$

The electric field constraints (6) along with the $U(1) \otimes U(1)$ Gauss law constraints (33) imply

$$k_+(L)k_-(L) = k_+(R)k_-(R). \tag{B.1}$$

On the other hand, the action of k_+k_- on a general $Sp(2, R)$ irrep. $|k, m\rangle$ is given by [21]

$$k_+k_-|k, m\rangle = (m - k)(m + k - 1)|k, m\rangle, \tag{B.2}$$

where $m = k + \rho$. In the present case the electric field constraint (B.1) and the eigenvalue equation (B.2) imply

$$(m(L) - k(L))(m(L) + k(L) - 1) = (m(R) - k(R))(m(R) + k(R) - 1). \tag{B.3}$$

As $k(L) = k(R) = \frac{1}{2}(p + q + 3)$, we get the unique solution of (B.3):

$$\rho_L(n, i) = \rho_R(n + i, i). \tag{B.4}$$

Therefore, in the prepotential Hilbert space $\mathcal{H}_p$ the left and the right $Sp(2, R)$ 'magnetic' quantum numbers are the same on every link.

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